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ON THE EXISTENCE OF AN INTEGRAL INVARIANT OF A SMOOTH DYNAMIC SYSTEM*

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The existence of an integral invariant with a smooth density for a dynamic system in a cylindrical phase space is considered. The well-known Krylov-Bogolyubov theorem guarantees the existence of an invariant measure for any system in a compact space (for a discussion of these topics see /1, 2/). But this measure is often concentrated in invariant sets of small dimensionality and in general is not an integral invariant with a summable density. For useful applications of ergodic theory, and in the theory of the Euler-Jacobi integrating factor, an invariant measure in the form of an integral invariant with smooth density is useful. Effective criteria for the existence of such measures in smooth dynamic systems are described. The general results are illustrated by examples from non-holonomic mechanics.

1. Formulation of the problem. Consider the cylindrical phase space $M^n = \mathbf{R}^k \times \mathbf{T}^{n-k}$ with coordinates $x_1, \ldots, x_n$, of which k are linear and $n-k$ angular. Let v be a smooth vector field in M^n; the corresponding differential equation is

$$x^{\cdot} = v(x) \tag{1.1}$$

We consider the existence for system (1.1) of the integral invariant

$$\operatorname{mes}(D) = \int_D f(x)\, d^n x \tag{1.2}$$

with smooth positive density $f: M^n \to \mathbf{R}$.

The criterion for the existence of integral invariant (1.2) is the Liouville equation $\operatorname{div}(fv) = 0$, which, since f is positive, can be rewritten as

$$w^{\cdot} = -\operatorname{div} v,\ w = \ln f \tag{1.3}$$

Clearly, w is a smooth function in M^n.

By the theorem on the rectification of trajectories, in a small neighbourhood of a non-singular point of system (1.2) there is an entire family of integral invariants. Thus it is worth considering the integral invariant problem either in the neighbourhood of a position of equilibrium, or in a sufficiently large domain of the phase space where the trajectories are reversible.

We know that the equation of motion of holonomic mechanical systems always have a natural invariant measure (the shape of the volume in the space of cotangent fiberings of the space of positions). It was pointed out in /3/ that non-holonomic systems may in general not have an invariant measure with an integrable density.

We will mention two examples of non-holonomic systems which will be used to illustrate our results.

1°. The problem of the rolling of a heavy rigid body over an absolutely rough horizontal plane. Chaplygin found the invariant measure in the case when the surface is bounded by a sphere and the centre of mass of the body is the same as its geometric centre /4/. An invariant measure can also be shown to exist when the rigid body has an axis of symmetry (either geometric or dynamic).

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2°. Suslov's problem on the inertial rotation of a rigid body about a fixed point with non-holonomic coupling: the projections of the angular velocity onto a direction fixed in the body vanish /5/. Let $Ox_1x_2x_3$ be a moving orthogonal system of axes, and p, q, r the projections of the angular velocity onto these axes: the matrix of inertia of the body is

$$\begin{Vmatrix} A & D & E \\ D & B & F \\ E & F & C \end{Vmatrix} \tag{1.4}$$

α, β, γ are the direction cosines of the fixed vertical relative to the x_1, x_2, x_3 axes. The equation of the non-holonomic coupling is taken to be $r = 0$. By a rotation of the x_1 and x_2 axes we can arrange to satisfy the equation $D = 0$. In the variables $p, q, \alpha, \beta, \gamma$, the equations of rotation of the body are /5/

$$Ap^{\cdot} = Epq + Fq^2, \quad Bq^{\cdot} = -Fpq - Ep^2$$
$$\alpha^{\cdot} = -q\gamma, \quad \beta^{\cdot} = p\gamma, \quad \gamma^{\cdot} = q\alpha - p\beta \tag{1.5}$$

It was shown in /3/ that these equations have an integral invariant if and only if $E = F = 0$. In this case the x_3 axis is proper for the matrix of intertia (1.4).

2. Condition of existence for the integral invariant.

Theorem 1. Let $x : \mathbf{R} \to M^n$ be the solution of system (1.1) with a compact closure of its trajectory. If system (1.1) has an integral invariant, there exists

$$\lim_{s\to\infty} \frac{1}{s} \int_0^s (\operatorname{div} v)_{x(t)}\, dt = 0 \tag{2.1}$$

Proof. Let $x(t) \in D_0$ where D_0 is a compact subdomain in M^n. In accordance with (1.3)

$$\lim_{s\to\infty} \frac{1}{s} \int_0^s \operatorname{div} v\, dt = \lim_{s\to\infty} \frac{w(x(0)) - w(x(s))}{s} = 0$$

since the continuous function w is lower and upper bounded in the set D_0.

Corollary. Let $x = 0$ be an equilibrium solution of the non-linear system $x^{\cdot} = Xx + \ldots$. If $\operatorname{tr} X \neq 0$, this system has no integral invariant with a smooth density in the neighbourhood of the point $x = 0$.

For, in this case $(\operatorname{div} v)_{x=0} = \operatorname{tr} X$. It remains to use (2.1) for the equilibrium solution $x(t) \equiv 0$.

It is interesting that the condition $\operatorname{tr} X = 0$ implies the preservation of the standard form of the volume in $\mathbf{R}^n$ by a phase flow of the linear system $x^{\cdot} = Xx$. Hence, if a linear system with constant coefficients has at least one integral invariant, it must have the standard invariant measure.

In the non-autonomous case the Liouville equation is $\partial f/\partial t + \operatorname{div}(fv) = 0$. If $f > 0$, then, putting $w = \ln f$, we again arrive at Eq.(1.3). We know that solutions $f(x, t)$ of the non-autonomous Liouville equation always exist. They are uniquely defined e.g., by the values of the density f at $t = 0$. Consequently, in this situation it is natural to consider the existence of integral invariants of a special kind. For instance, let $x^{\cdot} = X(t)x + \ldots$ be a non-linear ω-periodic system with the trivial solution $x(t) \equiv 0$. It can be shown that the necessary condition for an integral invariant with an ω-periodic density to exist is

$$\int_0^\omega \operatorname{tr} X(t)\, dt = 0$$

We know that the exponent on the right-hand side of this equation is equal to the product of the multipliers of the linearized system.

For the equilibrium positions of non-holonomic systems, the sum of the characteristic numbers is equal to zero. The necessary condition for an integral invariant to exist is thus satisfied. This is not the case, however, for stationary motions (relative equilibria).

Let us give some examples.

1°. When a heavy rigid body rolls on a rough horizontal plane there are stationary motions when one of the central axes of inertia is vertical, and the body rotates with constant angular velocity while touching the plane at the same point O. We can conclude from the form of the characteristic equation /6/ that the sum of the characteristic numbers is proportional (with a non-zero coefficient) to $\sin\alpha\cos\alpha$, where α is the angle between the principal directions of the body surface at the point O and the other two horizontal central axes of inertia. Thus, if these axes do not coincide, the equations of rolling do not have an integral invariant. Non-coincidence of the dynamic and geometric axes is a typical feature of so-called celtic stones, see /6/.

2°. The equations of motion in Suslov's problem (1.5) have a stationary solution, for which

$$p = -kF,\ q = kE,\ \alpha = lp,\ \beta = lq,\ \gamma = 0 \tag{2.2}$$

Here, k and l are constants such that $(kl)^2(E^2+F^2)=1$ The solution (2.2) exists only under the obvious condition $E^2+F^2 \neq 0$.

The matrix trace of the linearized system is $k(E^2/A+F^2/B)$. Hence, by the corollary to Theorem 1, the integral invariant for system (1.5) exists only under the condition $E=F=0$ This conclusion is reached in /3/ from geometric arguments.

3. Integral invariant in the neighbourhood of an equilibrium position.

Consider the non-linear system

$$x_s^{\cdot} = \sum \lambda_r^{(s)} x_r + \frac{1}{2}\sum a_{ij}^{(s)} x_i x_j + \ldots \tag{3.1}$$

The question of the existence of an integral invariant reduces to the question of the solvability of (1.3). We put

$$w = w_0 + (a, x) + \ldots;\ w_0 \in \mathbf{R},\ a \in \mathbf{R}^n$$

We will calculate the divergence of the right-hand side of system (3.1):

$$-\operatorname{div} v = \operatorname{tr}\Lambda + (b, x) + \ldots$$

$$\Lambda = \|\lambda_r^{(s)}\|,\quad b = (b_1, \ldots, b_n)^T,\quad b_j = \sum_{s=1}^n a_{sj}^{(s)}$$

In the present case Eq.(1.3) has the explicit form

$$(a, \Lambda x) + \ldots = -\operatorname{tr}\Lambda - (b, x) - \ldots \tag{3.2}$$

Hence we obtain the series of equations: $\operatorname{tr}\Lambda = 0$, $\Lambda^T a = b, \ldots$ The first was obtained above in Sect.2. We put $X = \Lambda^T, Y = \|X, b\|$. The matrix Y has dimensions $n \times (n+1)$.

Theorem 2. If $\operatorname{rank} X < \operatorname{rank} Y$, system (3.1) does not have an integral invariant in the neighbourhood of the point $x = 0$.

For, we always have $\operatorname{rank} X \leqslant \operatorname{rank} Y$ and the equation is the condition for the linear system $\Delta^T a = b$ to be solvable with respect to a. This condition certainly holds if Λ is a non-degenerate matrix.

Notice that the constant w_0 in the Maclaurin expansion of w can be regarded as arbitrary. This corresponds to the fact that the density of the invariant measure is defined apart from a positive constant factor. The condition for Eq.(3.2) to be solvable to a first approximation is that the ranks of X and Y should be equal. If this is so, we can find (possibly non-uniquely) the linear terms in x in the expansion of w. On equating coefficients of the terms of second degree in Eq.(3.2), we obtain a linear system of algebraic equations for the quadratic terms in the expansion of w. The components of the vector a appear linearly in this system. Its solvability condition is the well-known condition for the Kronecker-Capelli ranks to be equal. Similarly, when analysing the solvability of Eq.(3.2) in the higher approximations, conditions arise on the ranks of matrices whose elements are expressible in terms of the coefficients of the right-hand sides of system (3.1).

As an application of Theorem 2, we again consider system (1.5) of Suslov's problem. Since no forces act on the rigid body, any position of it is an equilibrium position. Let $\alpha_0, \beta_0, \gamma_0$ be the direction cosines in the equilibrium state. The divergence of the vector field (1.5) is equal to $qE/A - pF/B$, so that $b = (-F/B, E/A, 0, 0, 0)^T$. We write the non-trivial non-zero part of the matrix Y as

$$\left\|\begin{array}{ccc|c} 0 & \gamma_0 & -\beta_0 & -F/B \\ -\gamma_0 & 0 & \alpha_0 & E/A \end{array}\right\|$$

The vertical line separates the elements of the matrix X. Let $\gamma_0 = 0$. Then $\operatorname{rank} X = 1$. If

$$E\beta_0/A - F\alpha_0/B \neq 0 \tag{3.3}$$

then $\operatorname{rank} Y = 2$. Let $E^2+F^2 \neq 0$. We can then choose α_0 and β_0 $(\alpha_0^2+\beta_0^2=1)$ in such a way that (3.3) holds. In this case, as we mentioned in Sect.1, the equations have no integral invariant.

4. The averaging principle.

We consider the systems of normal form commonly encountered in applications:

$$I_k^{\cdot} = \varepsilon F_k(I, \varphi) + \ldots,\quad \varphi_s^{\cdot} = \omega_s(I) + \varepsilon G_s(I, \varphi) + \ldots \tag{4.1}$$

Here, $I = (I_1, \ldots, I_m)$ are Cartesian coordinates in $\mathbf{R}^m$, $\varphi = (\varphi_1, \ldots, \varphi_n)$ is the set of angular coordinates in the n-dimensional torus $\mathbf{T}^n$, ε is a small parameter. The functions $F_s, G_s, \ldots$ are assumed to be 2π-periodic in the angular variables $\varphi_1, \ldots, \varphi_n$. With $\varepsilon = 0$ system (4.1) obviously has a family of invariant measures, whose densities are arbitrary smooth positive functions of the variables $I_1, \ldots, I_m$.

We examine the case when system (4.1) has an integral invariant with density in the form of the series

$$f_0(I, \varphi) + \varepsilon f_1(I, \varphi) + \ldots \tag{4.2}$$

with coefficients f_α $(\alpha = 0, 1, \ldots)$ which are smooth and single-valued in $\mathbf{R}^m \times \mathbf{T}^n$. Clearly, $f_0 > 0$.

The undisturbed system (4.1) is easily integrated; the variables I, which are first integrals, number the invariant toruses, filled by conditionally periodic motions with frequencies $\omega_1, \ldots, \omega_n$. The invariant torus $I = I_0$ is called non-resonant if $(\omega(I^0), k) \neq 0$ for all integer-valued vectors $k \neq 0$. The phase trajectories everywhere densely fill the non-resonant toruses. We call the undisturbed system non-degenerate if the non-resonant toruses everywhere densely fill the phase space $\mathbf{R}^m \times \mathbf{T}^n$.

Theorem 3. Assume that the undisturbed system is non-degenerate and that Eqs.(4.1) have an integral invariant with density (4.2). Then the averaged system

$$\dot{J}_k = \varepsilon \bar{F}_k(J), \quad k = 1, \ldots, m \tag{4.3}$$

has an integral invariant with density $\bar{f}_0$.

The bar denotes as usual the result of applying the averaging operator with respect to the angular variables φ. Passage from the complete system (4.1) to the averaged system (4.3) is by the standard device of perturbation theory. We note a corollary of Theorem 3: if $k = 1$ and the function $\bar{F}$ has an isolated zero, then the complete system (4.1) has no integral invariant with density in the form of series (4.2).

Proof of Theorem 3. Since $f_0 > 0$, the function $w = \ln f$ can likewise be written in the neighbourhood of every invariant torus of the undisturbed system with small values of ε as a series in powers of ε: $w = w_0(I, \varphi) + \varepsilon w_1(I, \varphi) + \ldots$. In the present problem Eq.(1.3) has the form

$$\sum \frac{\partial w_0}{\partial \varphi_s} \omega_s + \varepsilon \left[\sum \frac{\partial w_0}{\partial I_k} F_k + \sum \frac{\partial w_1}{\partial \varphi_s} \omega_s \right] + \ldots = -\varepsilon \left[\sum_k \frac{\partial F_k}{\partial I_k} + \sum_s \frac{\partial G_s}{\partial \varphi_s} \right] + \ldots$$

Equating coefficients of like powers of ε, we obtain a chain of equations for finding successively $w_0, w_1, \ldots$:

$$\sum \frac{\partial w_0}{\partial \varphi_s} \omega_s = 0 \tag{4.4}$$

$$\sum \frac{\partial w_0}{\partial I_k} F_k + \sum \frac{\partial w_1}{\partial \varphi_s} \omega_s = -\sum \frac{\partial F_k}{\partial I_k} - \sum \frac{\partial G_s}{\partial \varphi_s} \tag{4.5}$$

It follows from (4.4) that w_0 is the integral of the undisturbed system. Since this system is non-degenerate, w_0 does not depend on the angular variables (cf. /7/, Chapter 5). On averaging (4.5) over φ, we arrive at the equation

$$\sum \frac{\partial w_0}{\partial I_k} \bar{F}_k = -\sum \frac{\partial \bar{F}_k}{\partial I_k}$$

Consequently, $f_0 = \exp w_0 > 0$ is likewise independent of φ and is the density of the invariant measure of the averaged system.

Theorem 3 can be extended to the case of the systems considered in /8/: with $\varepsilon = 0$ the phase space is fibered into integral manifolds. The non-degeneracy of the undisturbed system means that the integral manifolds in which the dynamic system is ergodic are everywhere dense.

5. Integral invariant of systems of normal form.

We indicate below more exact conditions for the existence of an integral invariant of system (4.1). We expand functions F_k and G_k in multiple Fourier series

$$F_k = \sum_\alpha F_\alpha^{(k)}(I) e^{i(\alpha, \varphi)}, \quad G_s = \sum_\alpha G_\alpha^{(s)}(I) e^{i(\alpha, \varphi)}; \quad \alpha \in \mathbf{Z}^n$$

We call the key set of system (4.1) the set of all points $I \in \mathbf{R}^m$ such that.

1) $(\omega(I), \xi) = \ldots = (\omega(I), \zeta) = 0$ with integer-valued vectors $\xi, \ldots, \zeta$,

2) $\operatorname{rank} X < \operatorname{rank} Y$, where

$$X=\left\|\begin{matrix}F_\xi^{(1)} & \ldots & F_\xi^{(m)}\\ \ldots & \ldots & \ldots\\ F_\zeta^{(1)} & \ldots & F_\zeta^{(m)}\end{matrix}\right\|, \qquad Y=\left\|X\left|\begin{matrix}\sum\frac{\partial F_\xi^{(k)}}{\partial I_k}+i\sum\xi_s G_\xi^{(s)}\\ \ldots\ldots\ldots\\ \sum\frac{\partial F_\zeta^{(k)}}{\partial I_k}+i\sum\zeta_s G_\zeta^{(s)}\end{matrix}\right.\right\|$$

Let us emphasize that linear independence of the vectors $\xi, \ldots, \zeta$, is not assumed in this definition.

Theorem 4. If the undisturbed system is not degenerate and the key set is not empty, then system (4.1) has no integral invariant with density $\Sigma f_s \varepsilon^s$.

The poof is by Poincaré's method (/7/, Chapter 5). We start from Eq.(4.5), in which w_0 is an unknown smooth function of the variables $I_1, \ldots, I_m$. We put

$$w_1=\sum W_\alpha(I)\, e^{i(\alpha,\varphi)}$$

Using Fourier's method, we obtain from (4.5) the series of equations

$$\sum\frac{\partial w_0}{\partial I_k}F_\alpha^{(k)}+i(\alpha,\omega)W_\alpha=-\sum\frac{\partial F_\alpha^{(k)}}{\partial I_k}-i\sum\alpha_s G_\alpha^{(s)};\quad \alpha\in\mathbf{Z}^n$$

Now let the point I belong to the key set. Putting α equal to $\xi, \ldots, \zeta$, we obtain a series of algebraic equations in the derivatives $\partial w_0/\partial I_1, \ldots, \partial w_0/\partial I_m$. The condition for solvability is that $\operatorname{rank} X = \operatorname{rank} Y$

Consider a simple example. Let I_0 be the equilibrium position of the vector field $\bar{F} = (F_0^{(1)}, \ldots, F_0^{(m)})$, and let the integer-valued vector $\alpha = 0$. Then, $\operatorname{rank} X = 0$, and $\operatorname{rank} Y = 1$, if, with $I = I_0$, the sum $\Sigma\partial F_0^{(k)}/\partial I_k$ is the divergence of the averaged system, linearized in the neighbourhood of the non-zero point I_0. In this case the key set contains the point I_0, so that Theorem 4 is applicable. Notice that we can also prove that there is no integral invariant in this example by using Theorem 3 and the corollary of Theorem 1.

Consider the case when Eqs.(4.1) have a first integral in the form of the series $H = H_0(I,\varphi)+\varepsilon H_1(I,\varphi)+\ldots$. In the conditions of Theorem 4, the function H_0 is independent of φ and at points of the key set

$$X\partial H_0/\partial I = 0 \tag{5.1}$$

Consequently, if a point of the key set is not critical for the function H_0, then rank X drops at least by unity from its maximum possible value.

For the proof we again use Poincaré's method. The function H_0 is the first integral of the undisturbed system, which is independent of φ, since the system is not degenerate. To a first approximation in ε, the identity $H^{\cdot}\equiv 0$ gives the relation

$$\sum\frac{\partial H_0}{\partial I_k}F_k+\sum\frac{\partial H_1}{\partial\varphi_s}\omega_s\equiv 0$$

Let $H_1=\Sigma_\alpha h_\alpha(I)\, e^{i(\alpha,\varphi)}$. Using Fourier's method, we arrive at the series of equations

$$\sum\frac{\partial H_0}{\partial I_k}F_\alpha^{(k)}+i(\alpha,\omega)h_\alpha=0$$

which is equivalent to (5.1) when I belongs to the key set.

6. Application to weakly non-holonomic systems.

Theorem 4 can be modified for the case of degenerate systems. Instead of arguments of a general kind, we shall consider an instructive example from non-holonomic mechanics. We consider the non-holonomic system with torus space of positions $\mathbf{T}^3=\{\varphi_1,\varphi_2,\varphi_3, \bmod 2\pi\}$ and non-holonomic coupling

$$\varphi_3^{\cdot}=\varepsilon(a_1\varphi_1^{\cdot}+a_2\varphi_2^{\cdot}) \tag{6.1}$$

The coefficients a_1 and a_2 are single-valued smooth functions in $\mathbf{T}^3$, and ε is a small parameter. With $\varepsilon=0$ we have motion over the two-dimensional torus $\varphi_3=\text{const}$. In this case the system is holonomic and the equations of motion have a natural invariant measure. For small values of $\varepsilon\neq 0$, the system deviates slightly from holonomic.

The condition for integrability of the coupling Eq.(6.1) to a first approximation in ε is

$$\partial a_1/\partial\varphi_2-\partial a_2/\partial\varphi_1=0 \tag{6.2}$$

We Fourier-expand a_1 and a_2 with respect to the variables φ_1 and φ_2:

$$a_s = \sum_k a_{k_1, k_2}^{(s)} \exp [i (k_1\varphi_1 + k_2\varphi_2)]$$

The coefficients $a_k^{(s)}$ are periodic functions in φ_3.

Condition (6.2) is equivalent to the following chain of relations connecting the Fourier coefficients:

$$k_2 a_{k_1, k_2}^{(1)} - k_1 a_{k_1, k_2}^{(2)} = 0 \qquad (6.3)$$

We now consider the dynamics of the non-holonomic system with coupling (6.1) and Lagrangian $L = (\varphi_1^{\cdot 2} + \varphi_2^{\cdot 2} + \varphi_3^{\cdot 2})/2$. Up to terms of order $o(\varepsilon)$, the equations of motion are

$$\varphi_1^{\cdot} = I_1, \ \varphi_2^{\cdot} = I_2, \ \varphi_3^{\cdot} = \varepsilon (a_1 I_1 + a_2 I_2), \ I_1^{\cdot} = I_2^{\cdot} = 0 \qquad (6.4)$$

Notice that the undisturbed system is degenerate.

We examine the case when system (6.4) has an integral invariant with density in the form of the series $f = f_0 + \varepsilon f_1 + \ldots$ with periodic coefficients in φ. We write Liouville's equation up to $o(\varepsilon)$:

$$I_1 \frac{\partial f}{\partial \varphi_1} + I_2 \frac{\partial f}{\partial \varphi_2} + \varepsilon \frac{\partial}{\partial \varphi_3} f (a_1 I_1 + a_2 I_2) = 0 \qquad (6.5)$$

Putting $\varepsilon = 0$, we obtain the equation for f_0. Using Fourier's method, we conclude that f_0 depends on I_1, I_2, and φ_3 (cf. /7/, Chapter 5).

To a first approximation in ε Eq.(6.5) has the form

$$I_1 \frac{\partial f_1}{\partial \varphi_1} + I_2 \frac{\partial f_1}{\partial \varphi_2} + \frac{\partial}{\partial \varphi_3} (a_1 I_1 + a_2 I_2) f_0 = 0 \qquad (6.6)$$

Again applying Fourier's method, we put $f_1 = \Sigma f_k^{(1)} e^{i(k, \varphi)}$. From (6.6) we obtain a series of equations for the Fourier coefficients. On the resonance lines $I_1 = \delta k_2$, $I_2 = -\delta k_1$ $(\delta \in \mathbf{R})$ we have

$$\frac{\partial}{\partial \varphi_3} g_k(\varphi_3) f_0 (\delta k_2, -\delta k_1, \varphi_3) = 0, \quad g_k = k_2 a_k^{(1)} - k_1 a_k^{(2)} \qquad (6.7)$$

In view of (6.3), this condition certainly holds when coupling (6.1) is integrable. The converse is false. For instance, let a_1 and a_2 be independent of φ_3. Condition (6.3) is not in general satisfied, though Eq.(6.7) has a solution in the form of the functions f_0 which are independent of φ_3.

Let us find the conditions for the infinite chain of Eqs.(6.7) to be solvable with respect to the functions f_0. From (6.7) we have

$$f_0 g_k = c_k(\delta) \qquad (6.8)$$

Two cases are possible: either the function g_k has a zero on the circle $\mathbf{T}^1 = \{\varphi_3 \bmod 2\pi\}$, or else it has no zeros. Since c_k is independent of φ_3, condition (6.3) must hold in the first case. Putting $\delta = 0$ in (6.8), we find that, in the second case, the ratios of g_k with different k are independent of φ_3. The conditions are sufficient for system (6.7) to be solvable.

For, with values of k corresponding to the first case, Eq.(6.7) is certainly satisfied. In the second case, for some vector k we put $f_0 = c/g_k$. We choose the sign of the constant c so that the function f_0 is positive. Since the ratios of g_k are constant, the function f_0 satisfies Eq.(6.7) with other values of k.

For the initial Eq.(6.6) to be solvable, extra conditions must be imposed: the free coefficients $a_0^{(1)}$ and $a_0^{(2)}$ either vanish, or the ratios $a_0^{(1)}/a_0^{(2)}$, $a_0^{(s)}/g_k$ $(s = 1, 2;\ g_k \neq 0)$ are independent of φ_3.

The results of analysing Eq.(6.6) can be stated geometrically. The set of all systems with Lagrangian $L = (\varphi_1^{\cdot 2} + \varphi_2^{\cdot 2} + \varphi_3^{\cdot 2})/2$ and with coupling (6.1) has the natural structure of a linear space (isomorphic to the space of pairs of smooth functions a_1 and a_2 in the three-dimensional torus). All systems which have an integral invariant (to a first approximation in ε) form a linear subspace $\mathbf{L}$. Similarly, the systems with integrable coupling (6.1) form a linear subspace $\mathbf{L}' \subset \mathbf{L}$. By analysing the solvability of Eqs.(6.7) we see that the dimensionality of the quotient space $\mathbf{L}/\mathbf{L}'$ is infinite. Thus systems with integrable coupling form a rare exception among the systems which have an integral invariant.

In conclusion, we consider the Chaplygin systems for which the functions a_1 and a_2 are independent of the variable φ_3. They also form a linear subspace, call it $\mathbf{L}''$. It can be shown that, for such systems, to a first approximation in ε all the conditions for the method of a reducing factor, ensuring the existence of an integral invariant /4/, to be applicable, are satisfied. For Chaplygin systems, $g_k = \text{const}$. Hence $\mathbf{L}'' \subset \mathbf{L}$. It can be shown that the dimensionality of the quotient space $\mathbf{L}/\mathbf{L}''$ is likewise infinite.

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ON THE MOTION OF CHAPLYGIN'S SLEDGE*

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The plane motion of Chaplygin's sledge is studied. In his original studies of this non-holonomic system, Chaplygin /1/ assume that the support plane is horizontal, and used a reduction factor to reduce the problem to the study of a Hamiltonian system with two degrees of freedom and one cyclical coordinate (i.e., a completely Liouville integrable system). A smooth reversible replacement of the phase variables is used below for the reduction. The motion is studied in detail by the methods of Hamiltonian mechanics, and the motion on an inclined plane is studied by the averaging method. The problem was earlier studied in /2-4/ for certain constraints on the position of the sledge centre of gravity. Chaplygin's equations of motion on an inclined plane were integrated in /3/ on the assumption that the centre of gravity lies on a line through the blade and perpendicular to the blade.

1. We consider the motion of a rigid body supported on a smooth inclined plane by a blade and two smooth roots (a "balanced" Chaplygin sledge), in a homogeneous field of gravity with acceleration g. The oriented space of this system is three-dimensional and can be written as the layer between two parallel planes R^2, opposite points of which are identified /3/, i.e., $M_0 = R^2 \times S^1$.

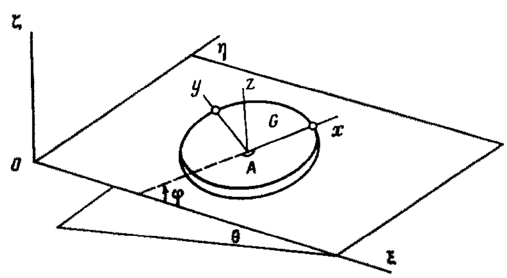

Fig.1

*Prikl.Matem.Mekhan.,51,4,546-551,1987